

ON FINITE DEFLECTIONS OF ANISOTROPIC LAMINATED ELASTIC PLATES†

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(Received 8 April 1985; in revised form 3 October 1985)

Abstract—The purposes of this paper are to (1) deduce a generalization of Von Karman's analysis, incorporating the effects of transverse shear deformations and of laminations, with the planes of the laminations not necessarily parallel to the plane of the plate; (2) further pursue the possibility of deriving plate equations from three-dimensional elasticity through use of a variational equation for displacements and transverse stresses; (3) consider the possibility of utilizing finite-elasticity variational formulations, including independent translational and rotational displacement components for the derivation of approximate two-dimensional results from three-dimensional theory.

INTRODUCTION

The considerations which follow have three different purposes. One purpose is to deduce a generalization of von Karman's analysis of plates, incorporating the effects of transverse shear deformation and of laminations, with the planes of the laminations not necessarily parallel to the midplane of the plate. A second purpose is the further pursuit of a possibility of deriving two-dimensional plate equations from three-dimensional elasticity through the use of a mixed variational equation for displacements and *transverse* stresses, in contrast to corresponding considerations involving displacements and *all* stresses, or displacements alone, or stresses alone. The third purpose is to consider the possibility of utilizing finite-elasticity variational formulations involving rotational displacements in addition to translational displacements for direct-methods derivations of approximate two-dimensional results from three-dimensional theory.

As regards the existence of related work we limit ourselves here to making reference to an analysis of the problem of *infinitesimal* deflections of laminated plates using a variational theorem for translational displacements and *transverse* stresses[3], and to a *formulation* of a variational theorem in finite elasticity for translational and rotational displacements and transverse stresses[2]. Other previous work on the derivation of linear two-dimensional plate equations by direct methods variational procedures is too well known to require enumeration in this place. As regards the relevant literature on variational theorems in finite elasticity, including in particular formulations by Frayes de Veubeke, Wempner, Bufler and Atluri, we here refer the reader to the list of references in Refs [1, 2].

THE VARIATIONAL EQUATION

Given a system of Cartesian coordinates x_i , we consider an elastic layer $-c \leq x_3 \leq c$, bounded by a cylindrical surface $f(x_1, x_2) = 0$. We assume, for ease of discussion of the essential aspects of what follows, that the faces $x_3 = \pm c$ of the layer are traction free and we do not here concern ourselves with the *details* of the traction distribution over the cylindrical boundary portion. Given this *three*-dimensional problem we develop our analysis to the point of establishing a system of *two*-dimensional differential equations, without the simultaneous derivation of a consistent system of two-dimensional boundary conditions, which we know to be possible.

† Based upon work supported by National Science Foundation Grant No. CEE-8213256.

We take as our point of departure a variational equation of the form

$$\delta \int \int \int [V(\gamma_{11}, \hat{\gamma}_{12}, \gamma_{22}; \hat{\tau}_{13}, \hat{\tau}_{23}, \tau_{33}) + \frac{1}{2}(\tau_{12} - \tau_{21})(\gamma_{12} - \gamma_{21}) + \gamma_{13}\tau_{13} + \gamma_{31}\tau_{31} + \gamma_{23}\tau_{23} + \gamma_{32}\tau_{32} + \gamma_{33}\tau_{33}] dx_3 dx_1 dx_2 = 0, \quad (1)$$

where

$$\hat{\gamma}_{12} = \gamma_{12} + \gamma_{21}, \quad \hat{\tau}_{13} = \frac{1}{2}(\tau_{13} + \tau_{31}), \quad \hat{\tau}_{23} = \frac{1}{2}(\tau_{23} + \tau_{32}). \quad (2)$$

In this the quantities τ_{ij} are components of stress which we have, in the present context[1, 2], designated as distinguished generalized Piola components. The quantities γ_{ij} are components of strain, which are *conjugate* to the τ_{ij} , inasmuch as the *virtual work* of the stresses τ_{ij} is of the form $\tau_{ij}\delta\gamma_{ij}$.

Expressions for the γ_{ij} , which we have earlier called generalized displacement gradient components[1, 2], can be written in the form

$$\gamma_{ij} = (\delta_{ik} + u_{k,i})\alpha_{jk} - \delta_{ij}, \quad (3)$$

in terms of the components of a displacement vector $\mathbf{u} = u_k \mathbf{e}_k$, and in terms of the components of an orthogonal unit vector triad $\mathbf{t}_j = \alpha_{jk} \mathbf{e}_k$ which is used in the stress vector representation $\tau_i = \tau_{ij} \mathbf{t}_j$.

With eqns (3) and the *partial* set of constitutive relations

$$\tau_{11} = \partial V / \partial \gamma_{11}, \quad \tau_{22} = \partial V / \partial \gamma_{22}, \quad \hat{\tau}_{12} = \partial V / \partial \hat{\gamma}_{12} \quad (4)$$

as constraint equations for the variational equation (1), the Euler equations of (1) come out to be the complementary set of constitutive relations, in the form

$$\hat{\gamma}_{13} = -\partial V / \partial \hat{\tau}_{13}, \quad \hat{\gamma}_{23} = -\partial V / \partial \hat{\tau}_{23}, \quad \gamma_{33} = -\partial V / \partial \tau_{33}, \quad (5)$$

together with equations of force and moment equilibrium of the form

$$(\alpha_{jk} \tau_{ij})_{,i} = 0, \quad (\delta_{in} + u_{n,i}) e_{mj k} \alpha_{mn} \tau_{ij} = 0. \quad (6a, b)$$

In order to derive (5) and (6a, b) we use the fact that the variations $\delta\tau_{13}$, $\delta\tau_{31}$, $\delta(\tau_{12} - \tau_{21})$ and δu_k are independent, and the the variations $\delta\alpha_{jk}$ are, as a consequence of the orthogonality equations $\alpha_{ik}\alpha_{jk} = \delta_{ij}$, expressible in the form $\alpha_{ik}\delta\alpha_{jk} = e_{ijm}\delta\chi_m$, in terms of three independent variations $\delta\chi_m$.

If we limit ourselves, as we shall do in what follows, to the range of *small* finite rotations, in terms of rotational parameters β_i , in such a way as to retain no more than first degree terms in β_3 and second degree terms in β_1 and β_2 then the quantities α_{jk} in (3) come out to be, in accordance with the analysis in Appendix B,

$$\begin{aligned} \alpha_{11} &= 1 - \frac{1}{2}\beta_1^2, & \alpha_{12} &= \beta_3 - \frac{1}{2}\beta_1\beta_2, & \alpha_{13} &= \beta_1, \\ \alpha_{21} &= -\beta_3 - \frac{1}{2}\beta_1\beta_2, & \alpha_{22} &= 1 - \frac{1}{2}\beta_2^2, & \alpha_{23} &= \beta_2, \\ \alpha_{31} &= -\beta_1, & \alpha_{32} &= -\beta_2, & \alpha_{33} &= 1 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\beta_2^2, \end{aligned} \quad (7)$$

and, if we further assume *small* finite strains, the quantities γ_{ij} come out to be

$$\begin{aligned} \gamma_{11} &= u_{1,1} + u_{3,1}\beta_1 - \frac{1}{2}\beta_1^2, & \gamma_{12} &= u_{2,1} - \beta_3 + u_{3,1}\beta_2 - \frac{1}{2}\beta_1\beta_2, \\ \gamma_{22} &= u_{2,2} + u_{3,2}\beta_2 - \frac{1}{2}\beta_2^2, & \gamma_{21} &= u_{1,2} + \beta_3 + u_{3,2}\beta_1 - \frac{1}{2}\beta_1\beta_2, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \gamma_{13} &= u_{3,1} - \beta_1, & \gamma_{31} &= u_{1,3} + \beta_1, \\ \gamma_{23} &= u_{3,2} - \beta_2, & \gamma_{32} &= u_{2,3} + \beta_2, \end{aligned} \tag{9}$$

$$\gamma_{33} = u_{3,3} - u_{1,3}\beta_1 - u_{2,3}\beta_2 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\beta_2^2, \tag{10}$$

with independent variations $\delta\beta_i$ in place of the variations $\delta\chi_m$.

For the derivation of an approximate two-dimensional plate theory we will make use of eqn (1) in the developed form

$$\begin{aligned} &\iint\iint\left[\frac{\partial V}{\partial\gamma_{11}}\delta\gamma_{11} + \frac{\partial V}{\partial\gamma_{22}}\delta\gamma_{22} + \left(\frac{\partial V}{\partial\hat{\gamma}_{12}} + \frac{\tau_{12} - \tau_{21}}{2}\right)\delta\gamma_{12} + \left(\frac{\partial V}{\partial\hat{\gamma}_{12}} - \frac{\tau_{12} - \tau_{21}}{2}\right)\delta\gamma_{21} \right. \\ &+ (\gamma_{12} - \gamma_{21})\delta\frac{\tau_{12} - \tau_{21}}{2} + \left(\frac{1}{2}\frac{\partial V}{\partial\tau_{i3}} + \gamma_{i3}\right)\delta\tau_{i3} + \left(\frac{1}{2}\frac{\partial V}{\partial\tau_{i3}} + \gamma_{3i}\right)\delta\tau_{3i} \\ &\left. + \left(\frac{\partial V}{\partial\tau_{33}} + \gamma_{33}\right)\delta\tau_{33} + \tau_{i3}\delta\gamma_{i3} + \tau_{3i}\delta\gamma_{3i} + \tau_{33}\delta\gamma_{33}\right] dx_3 dx_1 dx_2 = 0. \end{aligned} \tag{11}$$

DERIVATION OF APPROXIMATE TWO-DIMENSIONAL THEORY

Given various known results concerning two-dimensional theories of sixth or higher order for shear-deformable plates, through use of the variational theorems for stresses or displacements for the case of infinitesimal deformations, and through use of the variational theorem for displacements in terms of Green strains for small finite deformations, we here concern ourselves in particular with two specific questions. The first of these concerns the effect of the occurrence of the rotational displacement variables β_i , alongside the translational displacement variables u_i , in the mixed variational theorem for finite deformations in eqn (1). Eventually, we hope to arrive at a conclusion that use of the β_i in conjunction with the u_i , in place of the u_i alone with the conventional variational theorem in terms of Green strains, brings with it clearly identifiable advantages. For the present we limit ourselves to placing on record the simplest such result which we have been able to obtain. The second question concerns the form in which the effect of the transverse shearing stresses will appear in the system of two-dimensional constitutive equations if we allow for such anisotropy in a laminated material as is possible when the faces of the laminations are not necessarily parallel to the faces $x_3 = \pm c$ of the plate.

We begin our derivation by stipulating the well-known translational displacement component approximations

$$(u_1, u_2) = (v_1, v_2) + (\psi_1, \psi_2)x_3, \quad u_3 = w, \tag{12}$$

with v_i, ψ_i and w being functions of x_1 and x_2 .

The second step in our derivation is the nonconventional introduction of analogous approximations for the rotational variables β_i . Given the appearance of the strain components in eqns (8) and (9), and given the nature of the approximations in (12) the simplest

appropriate choice of the β_i is thought to be

$$(\beta_1, \beta_2) = (\phi_1, \phi_2), \quad \beta_3 = \omega + \phi_3 x_3, \quad (13)$$

with ϕ_i and ω being functions of x_1 and x_2 .

The third step consists in choosing approximations for the stress components τ_{i3} and τ_{3i} , and for the reactive stress quantity $\tau_{12} - \tau_{21}$. In this account we are limiting ourselves to the simplest possible rational approximations for these components by setting

$$(\tau_{i3}, \tau_{3i}) = (Q_{i3}, Q_{3i})/2c, \quad i = 1, 2, \quad (14)$$

$$\tau_{33} = 0, \quad \tau_{12} - \tau_{21} = 0. \quad (15)$$

In writing (14) and (15) we have avoided making the further approximations which would consist in stipulating $Q_{i3} = Q_{3i}$, and we note that for problems involving distributed transverse surface loads we should replace the approximation $\tau_{33} = 0$ by an expression accounting for these surface loads. We note further that with the assumed general character of anisotropy and nonhomogeneity in thickness direction, and with the way in which the stresses τ_{i3} and τ_{3i} appear in the variational equation (11), no other one-term approximations for these stresses would *generally* be superior to the ones made in (14).

Having now (12) to (15) we take account of the fact that the assumptions (15) reduce the variational equation (11) to the abbreviated form

$$\begin{aligned} \iiint \left[\frac{\partial V}{\partial \gamma_{11}} \delta \gamma_{11} + \frac{\partial V}{\partial \gamma_{22}} \delta \gamma_{22} + \frac{\partial V}{\partial \hat{\gamma}_{12}} \delta \hat{\gamma}_{12} + \left(\frac{1}{2} \frac{\partial V}{\partial \hat{\epsilon}_{13}} + \gamma_{i3} \right) \delta \tau_{i3} \right. \\ \left. + \left(\frac{1}{2} \frac{\partial V}{\partial \hat{\epsilon}_{i3}} + \gamma_{3i} \right) \delta \tau_{3i} + \tau_{i3} \delta \gamma_{i3} + \tau_{3i} \delta \gamma_{3i} \right] dx_3 dx_1 dx_2 = 0, \quad (16) \end{aligned}$$

and that, as a consequence, the strain component γ_{33} in (10) disappears from our consideration, while the strain components γ_{12} and γ_{21} occur only as the combination $\hat{\gamma}_{12}$. With this, and with (12) and (13), we now write

$$(\gamma_{11}, \gamma_{22}, \hat{\gamma}_{12}) = (\epsilon_{11}, \epsilon_{22}, \hat{\epsilon}_{12}) + (\kappa_{11}, \kappa_{22}, \hat{\kappa}_{12}) x_3, \quad (17)$$

where

$$\epsilon_{11} = v_{1,1} + w_{,1} \phi_1 - \frac{1}{2} \phi_1^2, \quad \epsilon_{22} = v_{2,2} + w_{,2} \phi_2 - \frac{1}{2} \phi_2^2,$$

$$\hat{\epsilon}_{12} = v_{1,2} + v_{2,1} + w_{,2} \phi_1 + w_{,1} \phi_2 - \phi_1 \phi_2, \quad (18)$$

$$\kappa_{11} = \psi_{1,1}, \quad \kappa_{22} = \psi_{2,2}, \quad \hat{\kappa}_{12} = \psi_{1,2} + \psi_{2,1}, \quad (19)$$

$$\gamma_{i3} = w_{,i} - \phi_i, \quad \gamma_{3i} = \psi_i + \phi_i. \quad (20)$$

The introduction of (17) and (14), in conjunction with an observation of the x_3 -independence of γ_{i3} and γ_{3i} in (20), and in conjunction with the defining relations

$$(N_{11}, N_{22}, N_{12}) = \int_{-c}^c \left(\frac{\partial V}{\partial \gamma_{11}}, \frac{\partial V}{\partial \gamma_{22}}, \frac{\partial V}{\partial \hat{\gamma}_{12}} \right) dx_3, \quad (21)$$

$$(M_{11}, M_{22}, M_{12}) = \int_{-c}^c \left(\frac{\partial V}{\partial \gamma_{11}}, \frac{\partial V}{\partial \gamma_{22}}, \frac{\partial V}{\partial \hat{\gamma}_{12}} \right) x_3 dx_3, \quad (22)$$

$$\gamma_i = -\frac{1}{2c} \int_{-c}^c \frac{\partial V}{\partial \tau_{i3}} dx_3, \tag{23}$$

reduces the triple integral relation (16) to a double integral relation, of the form

$$\iint \left[N_{11} \delta \varepsilon_{11} + N_{12} \delta \varepsilon_{12} + N_{22} \delta \varepsilon_{22} + M_{11} \delta \kappa_{11} + M_{12} \delta \kappa_{12} + M_{22} \delta \kappa_{22} + Q_{i3} \delta \gamma_{i3} \right. \\ \left. + Q_{3i} \delta \gamma_{3i} + \left(\gamma_{3i} - \frac{1}{2} \gamma_i \right) \delta Q_{3i} + \left(\gamma_{i3} - \frac{1}{2} \gamma_i \right) \delta Q_{i3} \right] dx_1 dx_2 = 0. \tag{24}$$

With $\delta \varepsilon_{11}$, $\delta \varepsilon_{22}$, $\delta \varepsilon_{12}$, $\delta \kappa_{11}$, $\delta \kappa_{22}$, $\delta \kappa_{12}$, $\delta \gamma_{i3}$ and $\delta \gamma_{3i}$ expressed in terms of δv_i , δw , $\delta \phi_i$ and $\delta \psi_i$ and their derivatives, eqn (24) implies the seven Euler equilibrium equations,

$$N_{11,1} + N_{12,2} = 0, \quad N_{12,2} + N_{22,2} = 0, \tag{25}$$

$$M_{11,1} + M_{12,2} = Q_{31}, \quad M_{12,1} + M_{22,2} = Q_{32}, \tag{26}$$

$$Q_{13,1} + Q_{23,2} + \phi_{1,1} N_{11} + (\phi_{1,2} + \phi_{2,1}) N_{12} + \phi_{2,2} N_{22} = 0, \tag{27}$$

$$Q_{31} - Q_{i3} + (w_{,1} - \phi_1) N_{11} + (w_{,2} - \phi_2) N_{12} = 0, \tag{28}$$

$$Q_{32} - Q_{23} + (w_{,1} - \phi_1) N_{12} + (w_{,2} - \phi_2) N_{22} = 0,$$

together with the four Euler constitutive relations

$$w_{,i} + \psi_i = \gamma_i, \quad 2\phi_i = w_{,i} - \psi_i, \tag{29}$$

where (29), with γ_i as in (23), complements the constraint constitutive relations (21) and (22). Altogether, we now have in (21), (22) and (25) to (29) a system of 17 equations for the 17 dependent variables $N_{ij} = N_{ji}$, $M_{ij} = M_{ji}$, Q_{i3} , Q_{3i} , ϕ_i , ψ_i , v_i and w .

Given that the transverse shear stress resultants Q_{i3} and Q_{3i} enter into the constitutive equations which are associated with the equilibrium equations (25) to (28) entirely through the combination

$$Q_i = \frac{1}{2} (Q_{i3} + Q_{3i}), \tag{30}$$

it suggests itself to rewrite (26) and (27), with the help of (28) and (25) as follows

$$M_{11,1} + M_{12,2} = Q_1 + \frac{1}{2} [(\phi_1 - w_{,1}) N_{11} + (\phi_2 - w_{,2}) N_{12}], \tag{31}$$

$$M_{21,1} + M_{22,2} = Q_2 + \frac{1}{2} [(\phi_1 - w_{,1}) N_{12} + (\phi_2 - w_{,2}) N_{22}],$$

$$Q_{1,1} + Q_{2,2} + \frac{1}{2} [(\phi_{1,1} + w_{,11}) N_{11} + (w_{,22} + \phi_{2,2}) N_{22}] + (\phi_{1,2} + \phi_{2,1} + 2w_{,12}) N_{12} = 0, \tag{32}$$

and to note that as a consequence of (29)

$$w_{,i} - \phi_i = \frac{1}{2} \gamma_i, \quad w_{,i} + \phi_i = -2\psi_i + \frac{3}{2} \gamma_i. \tag{33}$$

One of the questions which remains to be resolved concerns the relative significance of the γ_r -terms in the equilibrium equations (31) and (32), relative to the significance of the Q_r -terms in the constitutive equations (21) and (22).

FINITE DEFLECTION EQUATIONS WITH LINEAR CONSTITUTIVE EQUATIONS

Given the general system of constitutive equations (21), (22), and (29) with (23), it is of particular interest to consider cases for which the semicomplementary energy function V contains the strains and stresses by way of second and first degree terms only. Omitting, for simplicity's sake, the effect of first degree terms we can then write, with suitable coefficients A, B, C ,

$$\begin{aligned}
 V = \frac{1}{2} & (A_{11}\gamma_{11}^2 + A_{22}\gamma_{22}^2 + A_{33}\hat{\gamma}_{12}^2) + A_{12}\gamma_{11}\gamma_{22} + A_{13}\gamma_{11}\hat{\gamma}_{12} + A_{23}\gamma_{22}\hat{\gamma}_{12} \\
 & - 2c^2(B_{11}\hat{\tau}_{13}^2 + B_{22}\hat{\tau}_{23}^2 + 2B_{12}\hat{\tau}_{13}\hat{\tau}_{23}) \\
 & + 2c(C_{11}\gamma_{11}\hat{\tau}_{13} + C_{12}\gamma_{11}\hat{\tau}_{23} + C_{21}\gamma_{22}\hat{\tau}_{13} + C_{22}\gamma_{22}\hat{\tau}_{23} + C_{31}\hat{\gamma}_{12}\hat{\tau}_{13} + C_{32}\hat{\gamma}_{12}\hat{\tau}_{23}). \quad (34)
 \end{aligned}$$

In connection with this form of the function V we note in particular that the B -terms represent the conventional transverse shear deformation effects, while the C -terms represent the effect of laminations which are not parallel to the plane of the undeflected plate. We further note the absence of τ_{33} -terms in (34), which would have to be present for cases without the first of the two approximative assumptions in (15).

For what follows it is of considerable notational convenience to write in (17)

$$\epsilon_{11} = \epsilon_1, \quad \epsilon_{22} = \epsilon_2, \quad \hat{\epsilon}_{12} = \epsilon_3; \quad \kappa_{11} = \kappa_1, \quad \kappa_{22} = \kappa_2, \quad \hat{\kappa}_{12} = \kappa_3, \quad (35)$$

and to write in (21), (22), (25), (31) and (32)

$$N_{11} = N_1, \quad N_{22} = N_2, \quad N_{12} = N_3; \quad M_{11} = M_1, \quad M_{22} = M_2, \quad M_{12} = M_3. \quad (36)$$

Therewith, and with (34), (14) and (30) we obtain from (21) to (23) as a system of *eight* two-dimensional constitutive equations

$$N_i = \epsilon_j \int A_{ij} dx_3 + \kappa_j \int A_{ij} x_3 dx_3 + Q_j \int C_{ij} dx_3, \quad (37)$$

$$M_i = \epsilon_j \int A_{ij} x_3 dx_3 + \kappa_j \int A_{ij} x_3^2 dx_3 + Q_j \int C_{ij} x_3 dx_3, \quad (38)$$

$$\gamma_i = Q_j \int B_{ij} dx_3 - \epsilon_j \int C_{ji} dx_3 - \kappa_j \int C_{ji} x_3 dx_3. \quad (39)$$

The associated equilibrium equations (25), (31) and (32) become, with the same change of notation

$$N_{1,1} + N_{3,2} = 0, \quad N_{3,1} + N_{2,2} = 0, \quad (40)$$

$$M_{1,1} + M_{3,2} = Q_1 - \frac{1}{4}(\gamma_1 N_1 + \gamma_2 N_3), \quad (41)$$

$$M_{3,1} + M_{2,2} = Q_2 - \frac{1}{4}(\gamma_1 N_3 + \gamma_2 N_2),$$

$$Q_{1,1} + Q_{2,2} = \left(\psi_{1,1} - \frac{3}{4} \gamma_{1,1} \right) N_1 + \left(\psi_{2,2} - \frac{3}{4} \gamma_{2,2} \right) N_2 + \left(\psi_{1,2} - \frac{3}{4} \gamma_{1,2} + \psi_{2,1} - \frac{3}{4} \gamma_{2,1} \right) N_3. \quad (42)$$

With the bending strains κ_i given in terms of the ψ_i , in accordance with (19), and with w_i and ϕ_i obtainable in terms of ψ_i and γ_i , in accordance with (29), it suggests itself to rewrite the expressions for the membrane strains ε_i in (29) with w_i and ϕ_i expressed in terms of ψ_i and γ_i . We find after an elementary calculation and upon neglecting second degree terms in γ_i

$$\begin{aligned} \varepsilon_1 &= v_{1,1} + \frac{1}{2} \psi_1^2 - \gamma_1 \psi_1, & \varepsilon_2 &= v_{2,2} + \frac{1}{2} \psi_2^2 - \gamma_2 \psi_2, \\ \varepsilon_3 &= v_{1,2} + v_{2,1} + \psi_1 \psi_2 - \gamma_2 \psi_1 - \gamma_1 \psi_2. \end{aligned} \quad (43)$$

As regards the solution of the system of differential equations as stated we here note as a possible procedure the following. We may use (39) to express the Q_i in terms of the v_j , ψ_j and γ_j . The introduction of these expressions in (37) and (38) then gives the N_i and M_i in terms of the v_j , ψ_j and γ_j . The subsequent introduction of this into the five equilibrium equations in (40) to (42) leaves us with five differential equations for six dependent variables. In order to have a sixth equation we must return to (29) and deduce from this as a further relation involving ψ_j and γ_j

$$(\psi_1 - \gamma_1)_{,2} = (\psi_2 - \gamma_2)_{,1}. \quad (44)$$

Given the existence of more practical reduction procedures for the special cases of the problem of finite deflections of non-shear-deformable plates, and for the problem of infinitesimal deflections of shear-deformable plates, it may well be that the above reduction procedure for the general case will turn out to be amenable to further improvements.

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APPENDIX A: DERIVATION OF THE SEMICOMPLEMENTARY ENERGY FUNCTION V

Given a complementary energy function $W = W(\tau_{11}, \hat{\tau}_{12}, \tau_{22}; \hat{\tau}_{13}, \hat{\tau}_{23}, \tau_{33})$ and the set of six constitutive relations $\gamma_{11} = \partial W / \partial \tau_{11}$, $\hat{\gamma}_{12} = \partial W / \partial \hat{\tau}_{12}$, etc., with the $\hat{\gamma}$ and $\hat{\tau}$ as in (2), we deduce a suitable semicomplementary energy function V upon inverting the subset of three constitutive equations

$$\gamma_{11} = \partial W / \partial \tau_{11}, \quad \hat{\gamma}_{12} = \partial W / \partial \hat{\tau}_{12}, \quad \gamma_{22} = \partial W / \partial \tau_{22}, \quad (A1)$$

in the form

$$(\tau_{11}, \hat{\tau}_{12}, \tau_{22}) = (\tau_{11}, \hat{\tau}_{12}, \tau_{22})(\gamma_{11}, \hat{\gamma}_{12}, \gamma_{22}; \hat{\tau}_{13}, \hat{\tau}_{23}, \tau_{33}), \quad (A2)$$

and upon then defining $V = V(\gamma_{11}, \hat{\gamma}_{12}, \gamma_{22}; \hat{\tau}_{13}, \hat{\tau}_{23}, \tau_{33})$ as

$$V = \gamma_{11} \tau_{11}(\gamma_{11}, \dots; \hat{\tau}_{13}, \dots) + \dots - W[\tau_{11}(\dots), \dots; \hat{\tau}_{13}, \dots]. \quad (A3)$$

Equation (A3) implies the subset of inverted constitutive equations

$$\tau_{11} = \partial V / \partial \gamma_{11}, \quad \hat{\tau}_{12} = \partial V / \partial \hat{\gamma}_{12}, \quad \tau_{22} = \partial V / \partial \gamma_{22}, \quad (A4)$$

and the introduction of (A3) into the variational equation

$$\delta \int [W(\tau_{11}, \dots, \hat{\tau}_{13}, \dots) - \gamma_{11}\tau_{11} - \gamma_{12}\tau_{12} - \dots - \gamma_{33}\tau_{33}] dv = 0, \tag{A5}$$

with γ_j as in (3) and with independent variations $\delta\tau_{ij}$, δu_k and $\delta\chi_m$, where $\delta\alpha_{jk} = e_{ijm}\alpha_{ik}\delta\chi_m$, transforms (A5) into the variational equation (1), with independent variations δu_k , $\delta\chi_m$, $\delta\tau_{13}$, $\delta\tau_3$, and $\delta(\tau_{12} - \tau_{21})$ [2].

For the case of a complementary energy function

$$W = W_0(\hat{\tau}_{13}, \hat{\tau}_{23}, \tau_{33}) + W_1(\hat{\tau}_{13}, \dots; \tau_{11}, \dots) + W_2(\tau_{11}, \hat{\tau}_{12}, \tau_{22}), \tag{A6}$$

where W_0 and W_2 are homogeneous of the second degree and W_1 is homogeneous of the first degree, the three constitutive equations (A1) are a system of three linear equations for τ_{11} , $\hat{\tau}_{12}$, τ_{22} , of the form

$$\frac{\partial W_2}{\partial \tau_{11}} = \gamma_{11} - \frac{\partial W_1}{\partial \tau_{11}}, \quad \frac{\partial W_2}{\partial \hat{\tau}_{12}} = \hat{\gamma}_{12} - \frac{\partial W_1}{\partial \hat{\tau}_{12}}, \quad \frac{\partial W_2}{\partial \tau_{22}} = \gamma_{22} - \frac{\partial W_1}{\partial \tau_{22}}, \tag{A7}$$

with solution functions which depend linearly on γ_{11} , $\hat{\gamma}_{12}$, γ_{22} and $\hat{\tau}_{13}$, $\hat{\tau}_{23}$, τ_{33} .

The introduction of these solution functions into the defining relation (A3), in the form

$$V = \left(\frac{\partial W_1}{\partial \tau_{11}} + \frac{\partial W_2}{\partial \tau_{11}} \right) \tau_{11} + \dots - W_0 - W_1 - W_2, \tag{A8}$$

and an observation of the homogeneity consequences

$$\tau_{11} \partial W_1 / \partial \tau_{11} + \dots = W_1, \quad \tau_{11} \partial W_2 / \partial \tau_{11} + \dots = 2W_2, \tag{A9}$$

gives as expression for V

$$V = W_2[\tau_{11}(\gamma_{11}, \dots; \hat{\tau}_{13}, \dots), \hat{\tau}_{12}(\dots), \tau_{22}(\dots)] - W_0(\hat{\tau}_{13}, \dots). \tag{A10}$$

In order to see the nature of the coefficients A , B , C in (34) in terms of the set of analogous coefficients in (A6) we write

$$W_2 = \frac{1}{2}(a_{11}\tau_{11}^2 + a_{22}\tau_{22}^2 + a_{33}\hat{\tau}_{12}^2) + a_{12}\tau_{11}\tau_{22} + a_{13}\tau_{11}\hat{\tau}_{12} + \dots, \tag{A11}$$

$$W_1 = c_{11}\tau_{11}\hat{\tau}_{13} + c_{12}\tau_{11}\hat{\tau}_{23} + c_{13}\tau_{11}\tau_{33} + \dots + c_{33}\hat{\tau}_{12}\tau_{33}, \tag{A12}$$

$$W_0 = \frac{1}{2}(b_{11}\hat{\tau}_{13}^2 + b_{22}\hat{\tau}_{23}^2 + b_{33}\tau_{33}^2) + b_{12}\hat{\tau}_{13}\hat{\tau}_{23} + b_{13}\hat{\tau}_{13}\tau_{33} + \dots \tag{A13}$$

Equations (A7) now become, with $a_{ij} = a_{ji}$,

$$\begin{aligned} a_{11}\tau_{11} + a_{12}\tau_{22} + a_{13}\hat{\tau}_{12} &= \gamma_{11} - c_{11}\hat{\tau}_{13} - c_{12}\hat{\tau}_{23} - c_{13}\tau_{33}, \\ a_{21}\tau_{11} + \tau_{22}\tau_{22} + a_{23}\hat{\tau}_{12} &= \gamma_{22} - c_{21}\hat{\tau}_{13} - c_{22}\hat{\tau}_{23} - c_{23}\tau_{33}, \\ a_{31}\tau_{11} + a_{32}\tau_{22} + a_{33}\hat{\tau}_{12} &= \hat{\gamma}_{12} - c_{31}\hat{\tau}_{13} - c_{32}\hat{\tau}_{23} - c_{33}\tau_{33}. \end{aligned} \tag{A14}$$

The introduction of the solutions τ_{11} , τ_{22} , $\hat{\tau}_{12}$ of the system (A14) into (A11) and the subsequent introduction of the resulting expression, together with W_0 from (A13), into (A10) gives V as the desired quadratic form in γ_{11} , γ_{22} , $\hat{\gamma}_{12}$ and $\hat{\tau}_{13}$, $\hat{\tau}_{23}$, τ_{33} . The case which is considered in the body of the text represents a slight practical simplification of the above, which results upon stipulating $c_{13} = 0$ in (A12) and $b_{13} = 0$ in (A13).

APPENDIX B: A PARAMETRIC REPRESENTATION FOR ROTATIONS

Given the set of six orthonormality conditions $\alpha_{ik}\alpha_{jk} = \delta_{ij}$ for the nine coefficients in the triad relations $t_j = \alpha_{jk}e_k$ it is to be expected that the α_{ij} can be expressed in terms of three unrestricted parameters. Of the various ways in which this can be done we here describe one, which is unsymmetrical in a fashion appropriate to its use for the analysis of plates.

We begin by introducing an angular rotation measure β_3 in the plane of the plate and a system of planar rotated unit vectors \hat{e}_i , in the form

$$\hat{e}_1 = e_1c_3 + e_2s_3, \quad \hat{e}_2 = e_2c_3 - e_1s_3, \tag{B1}$$

where $c_j \equiv \cos \beta_j$ and $s_j \equiv \sin \beta_j$.

Subsequent to this we introduce two angular measures β_1, β_2 for rotation out of this plane by writing

$$ht_1 = \hat{e}_1c_1 + e_3s_1 - \hat{e}_2g, \quad ht_2 = \hat{e}_2c_2 + e_3s_2 - \hat{e}_1g, \tag{B2}$$

where $h = (1 + g^2)^{1/2}$ and $g = s_1s_2/(c_1 + c_2)$, so as to have $t_i \cdot t_j = \delta_{ij}$.

Having t_1 and t_2 we write $t_3 = t_1 \times t_2$. With (B2) this makes

$$h^2 t_3 = e_3(c_1 c_2 - g^2) - \hat{e}_1(s_1 c_2 + s_2 g) - \hat{e}_2(s_2 c_1 + s_1 g). \quad (\text{B3})$$

The introduction of (B1) into (B2) and (B3) leads to the following expressions for the coefficients α_{ij} .

$$\begin{aligned} h\alpha_{11} &= c_3 c_1 + s_3 g, & h\alpha_{12} &= s_3 c_1 - c_3 g, & h\alpha_{13} &= s_1, \\ h\alpha_{21} &= -s_3 c_2 + c_3 g, & h\alpha_{22} &= c_3 c_2 - s_3 g, & h\alpha_{23} &= s_2, \\ h^2\alpha_{31} &= s_3 s_2 c_1 - c_3 s_1 c_2 + s_3 s_1 g - c_3 s_2 g, \\ h^2\alpha_{32} &= -s_3 s_1 c_2 - c_3 s_2 c_1 - s_3 s_2 g - c_3 s_1 g, \\ h^2\alpha_{33} &= c_1 c_2 - g^2. \end{aligned} \quad (\text{B4})$$

Retention of no more than second degree terms in β_1 and β_2 and first degree terms in β_3 in (B4) results in the approximate relations in (7).